PLANCKS UK and Ireland Preliminaries 2021

The University of Kent
20th February 2021

This exam contains 13 pages (including this cover page) and 10 exercises.
You are required to show your work on each problem on this exam. The following rules apply:

- This paper consists of **10 problems, each worth a total of 10 marks**. For questions with multiple parts, the subdivisions of marks are indicated.
- The exam will **start at 10:30 am**, there will be a 30 minute break half way through. Participants have a total working time of **four hours** to complete the paper.
- There will be a 30 minute period after the exam when you can submit your answers via the submission folder on the PLANCKS website (www.plancks.uk/kent-exam). **Submissions after 3:30 pm will not be marked.**
- Organise your work when scanning your answer sheets; indicate at the top of each page your team name, the question number, and page number for that question. **Please upload the answer to each question as a separate file.**
- Each team will be in a separate breakout room for the duration of the exam. We require **at least one member of your team to be in the Zoom meeting at all times.**
- When a problem is unclear, a participant can ask, via Zoom, for a clarification. If the response is relevant to all teams, it will be provided to the other teams.
- The use of hardware (including phones, tablets, etc.) and external sources (including textbooks, non-team members, the internet etc.) is not approved. Scientific, non programmable calculators, watches and medical equipment are allowed. The use of computers and electronic devices should be limited to receiving/uploading the questions and communicating with your team and the invigilators. You may use an electronic notepad device to write your answers on but please do not attempt to write your answers in LaTeX.
- If it is brought to the invigilators' attention that a team has been cheating or breaking the rules they will be disqualified. Disqualification can happen post submission.
- In situations to which no rule applies, the committee will rule on the matter.

A Party at Isaac's House

Question 1: Prologue – Where Our Host Discovers Gravity

In his quest to have an epic house party, Isaac bakes a cake and has an interesting discovery about his universe's gravity.

Question 2: The First Guests Arrive

Beans on toast and odd shaped glasses, this night is off to a wild start when Blaise and Christaan arrive.

Question 3: Coming or Going?

Isaac tries not to burn the beans while his guests mingle, Emmy comes across Erwin in the corner who asks for her help.

Question 4: Electrostatics by Conformal Mapping

A failed throw ends with a complex analysis of the damage.

Question 5: Parastatistics

Whose statistics are better? Will it end with a fight?

Question 6: Condensed Matter

Lev is eating all the party food.

Question 7: A Rotating Glass of Mead

A spilled drink ruins the party tunes.

Question 8: A Harmonious Synthesiser

Seventeenth century mathematician and Australian pop legend sing 'shake it off' with a triangle guy.

Question 9: Bernoulli's Airship

It is the closest thing to Eurovision they have in this universe.

Question 10: Epilogue - Where Our Host Tries to go to Bed

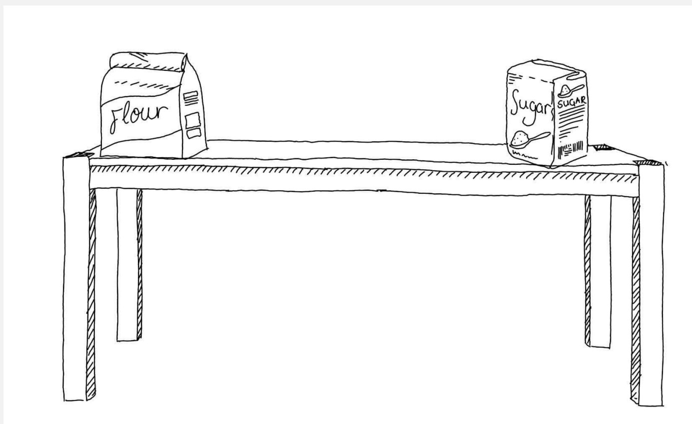
What do one-dimensional lattices and an infinite set of stairs have in common? Both impossible to directly navigate after a party...

1 Prologue – Where Our Host Discovers Gravity

Isaac is preparing to cook a cake for his dinner party later that week. Like any good physicist, he has polished his work counter until it is perfectly level and completely frictionless.

He bought the ingredients for the cake, 1kg of flour and 1kg of sugar at the point mass food store. He leaves them at rest on his counter exactly one metre apart. The next day he finds that the flour and sugar have moved towards each other due to their gravitational influence on each other.

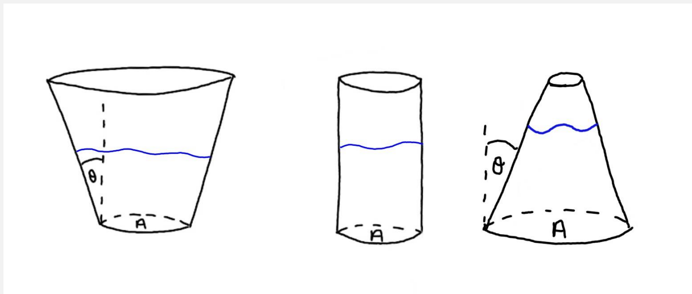
He times carefully that it takes 26 hours and 42 minutes for them to meet in the middle.



(10 marks) Two point masses, each of 1 kg start at rest a distance of 1m apart. The only force on them is their mutual gravitational attraction $F = -GMm/r^2$. If it takes 26 hours and 42 minutes for the two 1 kg masses to meet in the middle, calculate the value of the gravitational constant G in Isaac's universe.

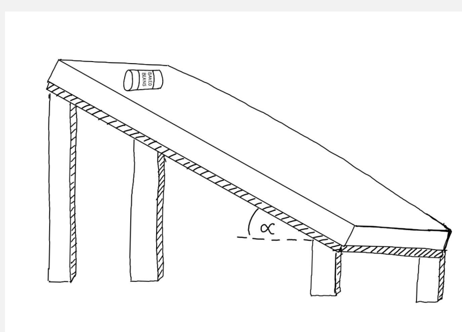
2 The First Guests Arrive

The day of the party has arrived and the first guest is Blaise. Isaac offers him a choice of drink – or rather the same drink in a choice of glass. All glasses have a circular bottom of the same area A, but in the first glass, the sides slope outwards at an angle θ, in the second glass the sides go straight up, while in the third glass the sides slope in at an angle θ.



(a) (4 marks) In order to be fair to the guests, Isaac thinks that the total integrated pressure on the bottom of each glass should be equal, but he and Blaise can't agree on how to achieve this. Isaac thinks pouring drink into each glass to the same height will do it, while Blaise thinks pouring the same volume in each glass is the way to go. Help them avoid a fight by solving the problem for them: If the straight glass is filled with a reference volume $V_2 = V$ of a drink, determine the volume of liquid to put in the other two glasses V_1 and V_3 so that the integrated pressure on the base of all three glasses is the same.

The next guest to arrive is Christaan. He is very excited to taste Isaac's main course, beans on toast. While Isaac is preparing this, Christaan notices that the table is not straight, making an angle α to the horizontal, and he idly starts rolling some cans of beans down the table. He notices that the full can of beans rolls faster than an empty can and wonders why that is. He is worried if he asks Isaac though that his toast will be burned.



(b) (6 marks) Help Christaan by calculating the acceleration felt by the empty and full can of beans as they roll down the table. You may assume the full can of beans is a cylinder of radius r, height h and constant density ρ, while the empty can of beans is a cylinder of radius r with all of its mass concentrated in the rim.

3 Coming or Going?

While Isaac is busy with the beans, Emmy spots Erwin in the corner of the room. "Thank goodness," Erwin says. I didn't know if I'd be here or not until you observed me. Erwin proceeds to tell Emmy about a curious problem he has been working on.

Consider a superposition of two plane wave solutions of the free-particle Schrodinger equation

$$\psi(x,t) = Ae^{i\theta_1(x,t)} + Be^{i\theta_2(x,t)},$$

where A and B are real positive constants and

$$\theta_n(x,t) = \frac{1}{\hbar}\left(p_n x - \frac{p_n^2 t}{2m}\right) + \gamma,$$

where the momenta p_n are also positive and γ is a constant.

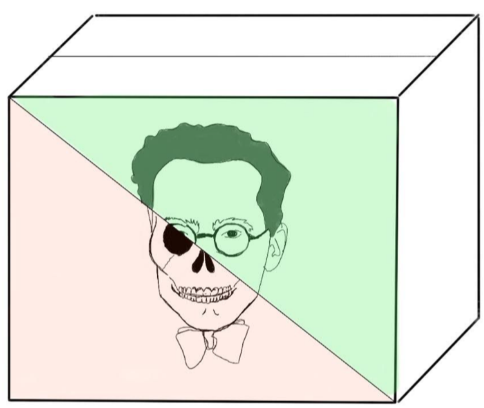
(a) (4 marks) Calculate the quantum mechanical current density given by

$$j(x,t) = \frac{i\hbar}{2m}\left(\psi(x,t)\frac{\partial\psi^*(x,t)}{\partial x} - \psi^*(x,t)\frac{\partial\psi(x,t)}{\partial x}\right).$$

(b) (2 marks) Why are your results odd (bizarre) if $A > B$ and $p_1 A < p_2 B$?

(c) (2 marks) Explain the origin of the phenomena in part (a).

(d) (2 marks) On the basis of this result what can be said about the same phenomena if it arises in a wavepacket composed of many plane waves rather than just two?



4 Electrostatics by Conformal Mapping

The first item of the night is broken as Charles-Augustin fails in his attempt to throw a bottle of cognac to Alessandro. After some cleaning up, Pierre-Simon walks over to inspect the hanging wires revealed from the accident.

The method of conformal mapping: establishes that for a given electrostatic potential $V(x, y)$ satisfying the Laplace equation in 2 dimensions, a new potential that also satisfies the Laplace equation can be obtained by taking an arbitrary conformal map $w(z)$ of the first one. This involves writing the two-dimensional coordinate x, y as single complex one $z = x + iy$. The conformal map is then any analytic function $w(z)$, meaning that the function should depend on the variable z but not its complex conjugate z^*. One first expresses the dependency on 2D coordinates in terms of the complex variable, $V(x, y) = V(z(x, y))$, and potentials can then be obtained by $W = V(w(z))$.

Consider two situations: in both cases there is a charged wire with charge density q running in the z-direction parallel to a perfectly conducting surface. In the first case, this surface is a plane $x = constant$, while in the second it is a parabola whose x-y coordinates in any plane of constant z satisfy the equation $x = \frac{y^2}{4\alpha^2}$.

(a) (2 marks) Calculate the electrostatic potential $V(x, y)$ generated when the wire is held a distance d from the plane.

(b) (5 marks) Find the electrostatic potential generated when the wire is aligned with the tip of the parabola with parameter α and held at a distance d from the tip.

(c) (3 marks) What is the enhancement of the field strength induced at the tip of the parabolic mirror, relative to the case of the plane mirror?

5 Parastatistics

Some of Isaac's guests are discussing statistics of fundamental particles.

Enrico and Paul are explaining that if you limit each state to have a maximum of one particle in it, a thermal distribution of particles will take the Fermi-Dirac form

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1},$$

where ϵ is the energy, μ is the chemical potential and k_B is Boltzmann's constant.

Satyendra and Albert on the other hand have figured out that if each state could have any number of particles in it, the thermal distribution takes the Bose-Einstein form

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}.$$

Isaac is listening to this conversation, and comments that there are a lot of other numbers between one and infinity.

(a) (6 marks) Consider a type of (fictional) particle where up to two particles may be characterised by a given set of quantum numbers. Find the resulting distribution function for such a particle, analogous to the Fermi-Dirac and Bose-Einstein distributions above.

(b) (2 marks) Find the limit of this distribution function for large temperatures, and briefly discuss its properties.

(c) (2 marks) Find the limit of this distribution function for small temperatures, and briefly discuss its properties.

6 Condensed Matter

On the other side of the room by the mini quiches, Lev beckons Enrico over to try some and discuss an article Isaac has left by the table on graphene and materials in d-dimensions.

A two dimensional material has an electronic band structure

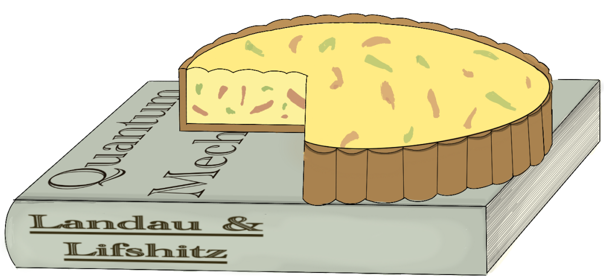
$$\epsilon(\underline{k}) = \epsilon_F \left(\frac{|\underline{k}|}{k_F} \right)^{\alpha},$$

where ϵ_F is the Fermi energy, k_F is the Fermi wavevector, and α is a parameter. The case $\alpha = 2$ corresponds to free electrons, while $\alpha = 1$ gives a linear spectrum as one would find, for example, in graphene. In this question, $\alpha > 0$ is considered an arbitrary parameter.

(a) (4 marks) Determine the density of states as a function of energy for a material with the above band-structure.

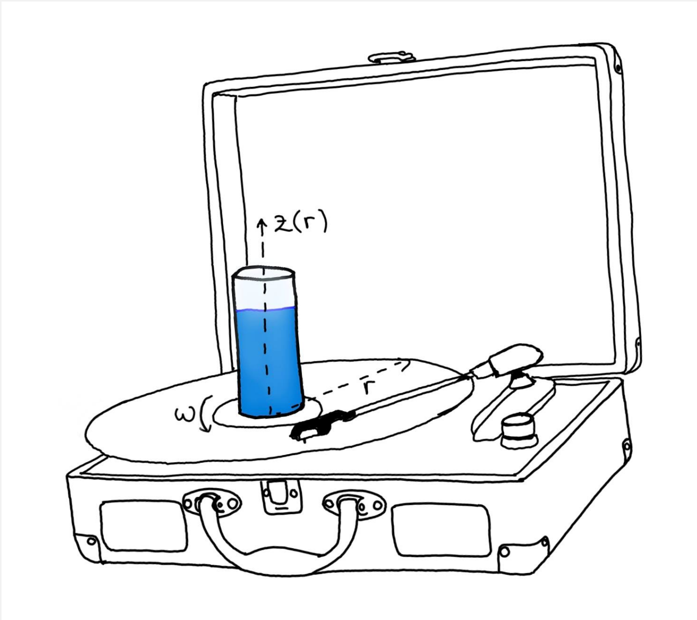
(b) (3 marks) Hence or otherwise determine the average electronic energy per particle E/N in the ground state.

(c) (3 marks) The previous steps have been for a two-dimensional material. Repeat the calculation of E/N for a d-dimensional material, where d is not necessarily integer.



7 A Rotating Glass of Mead

Isaac is very excited when his friend Kylie arrives. Kylie has brought a record of her latest music, which Isaac places on his turntable. Blaise who is close by, absentmindedly places his drink in the middle of the turntable too. After a bit of splashing around, the surface settles into a nice shape. Isaac thinks he's seen this shape before.

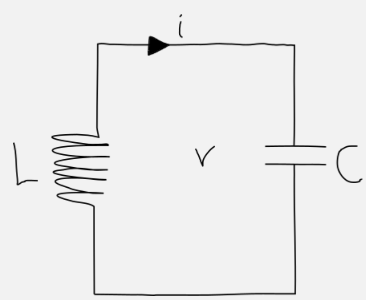


(10 marks) An open, circularly symmetric vessel containing liquid rotates freely about the vertical axis (z axis) through the centre of circular symmetry. The surface of the liquid will correspond to a surface of revolution generated by the curve $z(r)$ where r is the radial distance from the vertical axis. Given an angular frequency of rotation ω, and taking $z = 0$ to be the height of the surface of the liquid at $r = 0$, what is the equation of the curve $z(r)$?

8 A Harmonious Synthesiser

Blaise's efforts to dry out the record failed, luckily Heinrich brought his synthesiser to the party. Isaac and Kylie's faces immediately light up and Isaac runs into the back to grab his karaoke machine.

Synthesizers are musical instruments that produce a tone from the oscillations of a voltage $V(t)$ in an electronic circuit. The simplest electronic circuit that will oscillate at a given frequency is the LC resonator, formed by a capacitor C and and inductor L connected in series:



However, these oscillations are pure harmonic waves with a single frequency f that depends on the capacitance C and inductance L of the two electronic components in the circuit. In contrast, most musical sounds contain more than one frequency.

(a) (7 marks) You are asked to create an electric circuit that produces a sound with two, rather than a single frequency. You have at your disposal the electronic components shown in the diagram, that is, one capacitor and one inductor. In addition, you also have one additional capacitor and one additional inductor of variable capacitance αC and inductance βL, respectively (α and β are the tuning parameters and can independently taken any value between 0 and 100).

Using the components mentioned above (and an unlimited supply of wires) draw a single closed loop circuit that would produce two tones simultaneously and justify why it does.

Background information: A semitone is is an interval corresponding to an increase in frequency by a given fixed *factor* such that 12 such increases make up an octave (a doubling in frequency). Thus the 13 notes making up an octave (where the 13th note has twice the frequency of the first note) are regularly-spaced in frequency *when placed on a logarithmic scale*. This way of arranging notes on a scale corresponds to the equal-tempered tuning of a contemporary instrument in the Western musical tradition.

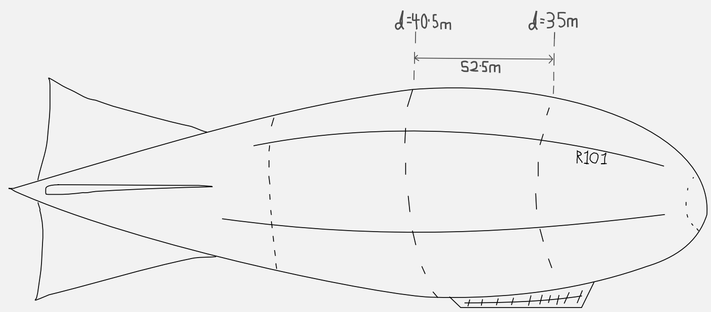
(b) (3 marks) Of course, superimposing two arbitrary frequencies will not, in general, lead to a harmonious sound. You are therefore asked to ensure that the two frequencies at which your circuit simultaneously oscillates form a *perfect fifth*, that is, they correspond to two notes separated by exactly seven semitones, or half-steps.

Give a combination of α and β that produces that effect.

9 Bernoulli's Airship

Having tired from the party and to calm his guests down from karaoke, isaac puts on a documentary about failed methods of air travel. Here is a summary of the segment on the R101 airship:

Rigid airships were used ostensibly during the beginning of the last century, as they carried the promise of 'safe' and reliable long distance air travel. One such early, though ill fated, rigid airship was the British built R101 which was first flown in 1929. The airship was filled with hydrogen and had a canvas fuselage reinforced with steel girders . It was only airborne for a year before it crashed, and was the biggest airship built until the similarly ill fated Hindenburg was built seven years later.



The characteristics of the R101 were:

- Length: 777 ft 0 in (236.8 m)
- Diameter: 131 ft 4 in (40.5 m)
- Height: 140 ft 0 in (42.67 m)
- Crew: 42 (final flight) (15 minimum)
- ρ_{air}= 1.29 kgm^{-3}
- ρ_H= 0.0899 kgm^{-3}
- ρ_{He}= 0.179 kgm^{-3}
- Empty weight: 257,395 lb (116,857 kg)

Naturally, the documentary did not calm the minds of the partygoers and they began asking questions regarding the R101 airship.

(a) (2 marks) What was the useful lift provided by the hydrogen used for the R101?

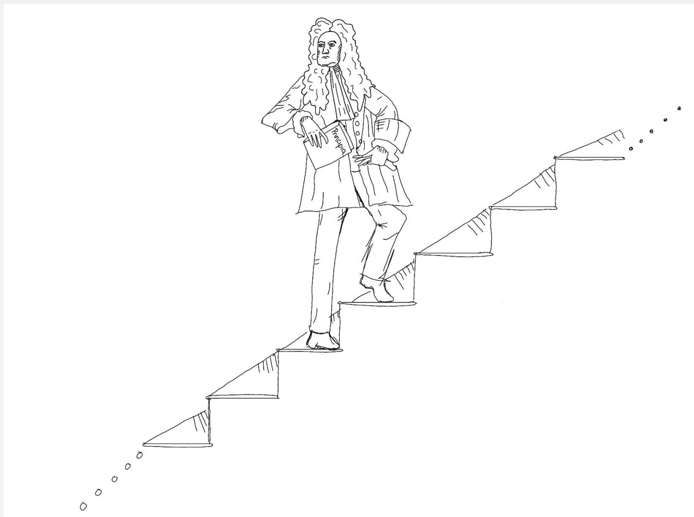
(b) (2 marks) These aircraft also saw extensive military use in the first world war, partially as (non-rigid) barrage balloons. They were designed to by a flying obstacle for the aircraft at the time, being tethered at around 1500m. How vulnerable were they to being shot down? Use both mathematical calculations and scientific reasoning in your answer.

(c) (2 marks)Both the R101 and the Hindenburg that followed resulted in fatal crashes after they caught fire. An alternative to the flammable Hydrogen would be to use Helium. What benefits/costs would this provide to the airship? Why was it not used?

(d) (4 marks) Another idea for an rigid airship is to use a vacuum instead of any gas to provide the lift. Calculate the lift provided if the R101 used a vacuum instead. Would this be a viable aircraft - use scientific reasoning to explain why and what it would take to make it so?

10 Epilogue - Where Our Host Tries to go to Bed

Isaac's guests have finally left and he plans to go to bed. The staircase in his house is infinite in extent, but he only has to go up 10 steps to get to his bedroom. Unfortunately, his mind is full of the joys of the party... and he keeps forgetting where he has come from and where he is going. On each step he goes up or down as a random process, with a slight bias of going down. Will he make it to bed?



This is analogous to a classical particle moving on an infinite one-dimensional lattice. In each time step, it either moves to the left with probability $p = 0.6$ or to the right with probability $1 - p = 0.4$.

(10 marks) What is the probability that the particle reaches 10 lattice steps to the right?

Questions contributed by:

Dr Sam Carr – Physics of Quantum Materials group, University of Kent

Dr James Kneller – IOP, London and Southeast Branch

Dr Gunnar Möller – Physics of Quantum Materials group, University of Kent

Dr Gavin Mountjoy – Materials for Energy and Electronics group, University of Kent

Dr Jorge Quintanilla – Physics of Quantum Materials group, University of Kent

Prof. Paul Strange – Emeritus, Physics of Quantum Materials group, University of Kent

With thanks to Anthony Quinlan for pressing 'recompile' on LaTeX, Holly Stokes-Geddes and Emily Sheehy for their fantastic illustrations.