PLANCKS Preliminaries 2022

UK, Ireland and Singapore
19th February 2022

This exam contains 18 pages (including this cover page) and 10 exercises.
You are required to show your work on each problem on this exam. The following rules apply:

- This paper consists of **10 problems, each worth a total of 10 marks**. For questions with multiple parts, the subdivisions of marks are indicated.
- The exam will **start at 10:30 am**, there will be a 30-minute break halfway through. Participants have a total working time of **four hours** to complete the paper.
- There will be a 30-minute period after the exam when you can submit your answers via the submission folder on the PLANCKS website (plancks.uk/exam). **Submissions after 3:30 pm will not be marked.**
- Organise your work when scanning your answer sheets; indicate at the top of each page your team name, the question number, and the page number for that question. **Please upload the answer to each question as a separate file.**
- Each team will be in a separate breakout room for the duration of the exam. We require **at least one member of your team to be in the Zoom meeting at all times.**
- When a problem is unclear, a participant can ask, via Zoom, for a clarification. If the response is relevant to all teams, it will be provided to the other teams.
- The use of hardware (including phones, tablets, etc.) and external sources (including textbooks, non-team members, the internet etc.) is not approved. Scientific, non-programmable calculators, watches and medical equipment are allowed. The use of computers and electronic devices should be limited to receiving/uploading the questions and communicating with your team and the invigilators. You may use an electronic notepad device to write your answers on, but please do not attempt to write your answers in LaTeX.
- If it is brought to the invigilators' attention that a team has been cheating or breaking the rules, they will be disqualified. Disqualification can happen post submission.
- In situations to which no rule applies, the committee will rule on the matter.

Life Aboard *Cheesecake*

You are a team of scientists on a space station called *Cheesecake* orbiting a star named *Blueberry*. Throughout this exam paper, you will tackle the daily problems that are faced by the scientists.

Question 1: The Stellar Engine Solution

Total destruction is 100 million years around the corner, can your civilisation survive?

Question 2: An Alternative Space Craft

A rebel group of scientists believes combustion is the way forward.

Question 3: What If... All Energy Was Electrical?

Plancks' Examinable Universe, invites you to journey to face the unknown and ponder the question...

Question 4: Refuelling with Hemisphere Hydrodynamics

Funnels, fuels and a friend from the Hundred Acre Wood.

Question 5: Simple Harmonic Mattresses

Beds are being worn out too quickly on the space station, can you find a solution (and a cause...)?

Question 6: The Firefly Dance

A bunch of biologists need some help, suppose we better help them with their fireflies.

Question 7: Bending Light in a Plane

What if we lived in two-dimensions?

Question 8: A Magnetic Spa Day

Relax, unwind and disorientate your dipoles.

Question 9: Droplets in SPACE

SPLISH, SPLASH, SPLOSH.

Question 10: Oscillators on a Train

Whip out your pocket oscillators, it's train time. Choo Choo.

1 The Stellar Engine Solution

An emergency meeting has been called, and your team has been summoned to the Omega Threat Level Situation Room.

A shocking discovery has been made, in 100 million years a foreign star will collide with your own and destroy your civilisation. You know that there is a 'safe zone' five light years away and decide that the best course of action is to move the star, rather than leaving the station. The government agrees to let you build a device which will move the star, but they want the costs and the impact on the civilisation's daily lives as well to remain as low as possible.

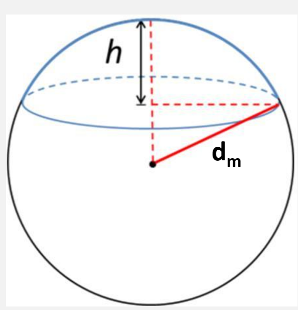
The device will use the momentum imparted by stellar photons to move the star and in turn your civilisation. The device will be made of *reflectonium*, an ultra-thin reflecting material, which your team will shape into a parabolic mirror and placed a distance d_m from the star. When the photons hit the mirror, they will be reflected having imparted momentum, hitting and leaving the surface.

Your team christens the device the *big-shiny-moving-star-inator* (or on Earth a *Shkadov thruster*), it is a statite meaning it will not orbit the star but remain at a fixed position (due to a balance of gravity and radiation pressure).

Useful data:

- Blueberry's peak spectrum wavelength, λ_{max}= 450 nm
- Blueberry's mass, $M = 4 \times 10^{31}$ kg
- Blueberry's radius, $R = 1.75 \times 10^{10}$ m
- Blueberry-to-inator distance, $d_m = 2 \times 10^{10}$ m
- Stefan-Boltzmann constant, $\sigma = 5.6704 \times 10^{-8}$ Wm^{-2}K^{-4}
- Wein's law, $\lambda_{max} T_{eff} = 2.898 \times 10^{-3}$ m K
- One light year $= 9.46 \times 10^{15}$ m
- One million years $= 3.1536 \times 10^{13}$ s

The surface area of a spherical cap is given by $A = 2\pi d_m h$, where h is the height (or depth) of the mirror.



(a) (8 marks) Workout the surface area of the *big-shiny-moving-star-inator* that:

  (i) (7 marks) meets the government's criteria.

  (ii) (1 mark) would take the shortest time to travel.

State any assumptions you have made to get to your answer.

The *big-shiny-moving-star-inator* is a statite: it will remain at a fixed position rather than orbiting around the star.
The radiation pressure is given below,

$$\langle P_{radiation} \rangle = \begin{cases} \frac{I}{c} & \text{Perfect absorber} \\ \frac{2I}{c} & \text{Perfect reflector,} \end{cases}$$

where I is the intensity at the mirror (flux) and c is the speed of light.

(b) (2 marks) Show that the position of the *big-shiny-moving-star-inator* will not affect its ability to be a statite.

2 An Alternative Space Craft

Another group of scientists on Cheesecake is investigating a different strategy of relocating to a star called Raspberry using a combustion engine.

The main idea is to develop the most energy-efficient fuel and produce a large amount of it to power the combustion engine. As a reference to the most efficient combustion reaction, scientists consider hydrogen, producing $q = 288$ kJmol^{-1}. Other reactions will provide a similar level of efficiency.

(a) (4 marks) Derive the relation between the achievable and fuel exhaust velocities. This relation should purely rely on momentum conservation in the absence of external force. Identify how the final velocity depends on the masses of the empty spacecraft (m_s) and fuel (m_f).

(b) (4 marks) Find the upper limit estimation of the achievable velocity in terms of the speed of light. It may help you know that the number of atoms in the observable universe is assumed to be $N = 10^{82}$.

(c) (2 marks) Find the combustion reaction energy q which allows achieving at least $0.5c$ final velocity.

3 What If... All Energy Was Electrical?

When attempting to build a new form of electrical generator, scientists at Cheesecake study the equivalence between the electron's mass and its electrical properties.

Any system energy is related to its mass by the Einstein relation $W = mc^2$. In this way, we can assign to, for example, electric field energy to its effective mass. Let us assume that the entire mass of an electron is "electric".

Useful data:

- electron mass $m = 9.1 \times 10^{-31}$ kg;
- electron charge $e = 1.6 \times 10^{-19}$ C;
- speed of light $c = 3 \times 10^8$ ms^{-1};
- Coulomb's law constant $k = 9 \times 10^9$ N m^{2}C^{-2};
- electric constant (vacuum permittivity) $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}.

(a) (5 marks) Determine the "electric" radius of the electron, assuming that the electron charge is evenly distributed over its surface.

In the most accurate experiments, Coulomb's law has been validated for radius as small as 10^{-16} m.

(b) (5 marks) Compare actual electron mass and effective "mass" of the electric field outside the sphere of 10^{-16} m radius.

4 Refuelling with Hemisphere Hydrodynamics

You have been brought in by the engineering department to look at the funnels used for refuelling the station's shuttles. You meet up with your friend Tigger who has already put on his orange overalls (that are always covered in black horizontal grease stains).

Tigger decided to learn some hydrodynamics, playing with a semi-spherical funnel. The funnel has an inner radius R, mass m and can be filled with a liquid of density ρ.

For all tasks, you may need the volume of a spherical cup formula (refer to figure 4 for notations):

- in terms of sphere radius and cup height $V = \frac{\pi h^2}{3}(3R - h)$;
- in terms of cup radius and height $V = \frac{1}{6}\pi h(3a^2 + h^2)$.

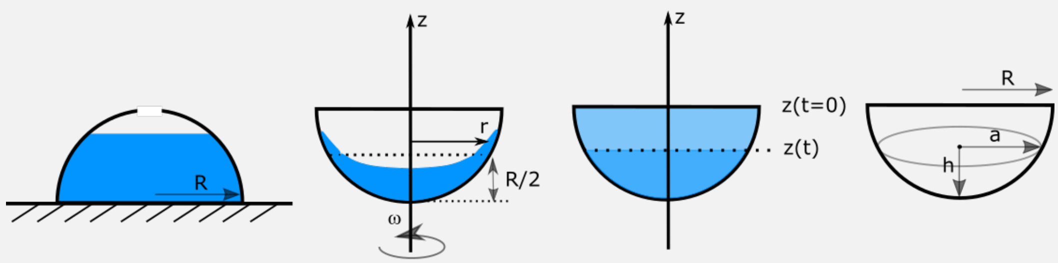


figure 1 **figure 2** **figure 3** **figure 4**

(a) (4 marks) Tigger places the funnel upside down on a rubber base, so, the connection between the funnel edge and the surface is hermetic. Through the hole in the top, Tigger pours the liquid in, measuring the total volume. At some moment, the hydrostatic pressure pushes the funnel up, and the liquid starts leaking. Identify the total liquid volume V when this happens. Assume that this volume is strictly less than the total funnel volume. Refer to figure 1.

(b) (4 marks) Another experiment Tigger performed is about the half-full funnel that spins with constant angular velocity ω. Your task is to find the equation of a liquid free surface that is formed by the balance between hydrostatic and centrifugal forces. As the system is axially symmetrical, it is sufficient to find the liquid level z as a function of radial coordinate r. Refer to figure 2.

(c) (2 marks) In the last experiment, Tigger heated up the full funnel with the heater of power P. Your task is to find how the level of the liquid will decrease with time. For simplicity, you can assume that the liquid is at its boiling temperature, and the unit mass heat of evaporation is q. Refer to figure 3.

5 Simple Harmonic Mattresses

The space station's maintenance team need your help, they have had complaints about the living quarters' mattresses. They would like you to research how different combinations of springs affect the bounciness of the mattresses.

This question is concerned with the oscillations of a single particle, mass m, moving on a smooth horizontal table under the influence of two or more perfect springs. Each spring obeys Hooke's law with stiffness k and has natural length l_0. In all diagrams, the system is in its equilibrium position.

For the system comprising just one spring with motion in the direction x, as shown in figure 1,

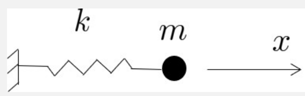


Figure 1: Single spring-mass system.

the period, τ, is, in terms of k and m, given by

$$\tau = 2\pi\sqrt{\frac{m}{k}}.$$

In the following parts you are asked to obtain the appropriate period, τ_i, in the form

$$\tau_i = \lambda_i \tau.$$

In each case the equilibrium position is defined by the length L with $L > l_0$.

(a) (2 marks) Consider the system with two springs as shown in figure 2.

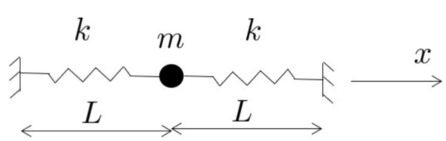


Figure 2: System with two springs along direction of motion.

Find the value of λ_1.

For the remaining parts, you should consider *small amplitude* oscillations in the appropriate direction x, i.e. $|x|/L << 1$. You may find it convenient to write $X = x/L$ and evaluate distances correct to first order in X, i.e. ignore terms X^2 and higher, and then write down the appropriate equation of motion.

(b) (3 marks) This part involves two springs, as shown in figure 3.

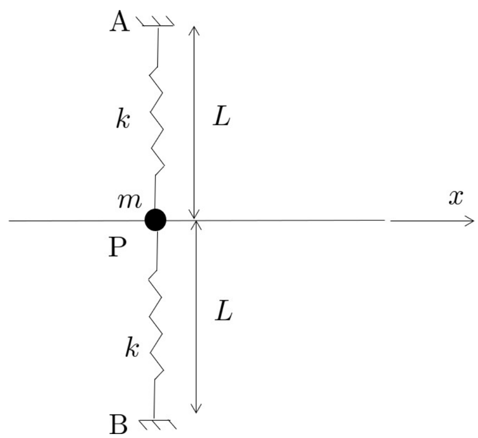


Figure 3: System with two springs perpendicular to direction of motion.

Find the value of λ_2.

(c) (2 marks) You may find the result in this section to be of help in sections (iv) and (v).

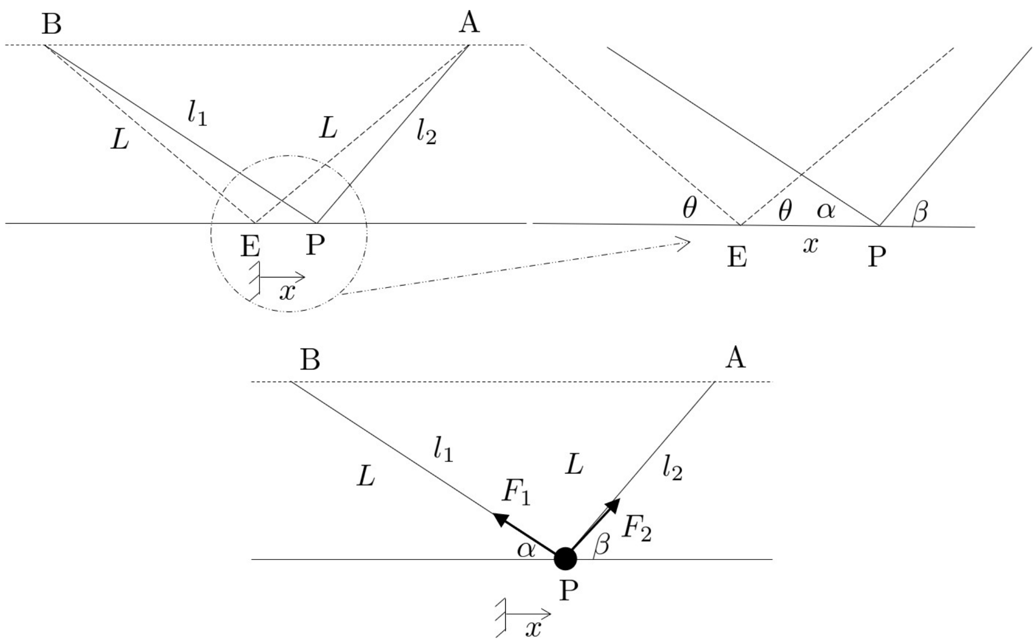


Figure 4: (a) Definition of l_1, l_2, α and β; (b) forces F_1 and F_2.

The situation shown in figure 4(a) shows a point, P, a *small* distance, x, from the point E. Show:
Either

$$l_1 = L\left(1 + X\cos\theta\right) + \mathcal{O}(X^2) \text{ and } \cos\alpha = \cos\theta + X\left(1 - \cos^2\theta\right) + \mathcal{O}(X^2).$$

Or

$$l_2 = L\left(1 - X\cos\theta\right) + \mathcal{O}(X^2) \text{ and } \cos\beta = \cos\theta + X\left(\cos^2\theta - 1\right) + \mathcal{O}(X^2).$$

Suppose that we have a symmetric system of springs such that the motion is always along the direction x as shown in figure 4. Two such springs are shown in figures 4(a) and (b). Using the results above, show that the magnitude of the force in each spring is given, to first order, by

$$F_1 = F_2 = kx\cos\theta$$

and that the tension force, F_1, and the compression force, F_2, yield a combined, first order, component in the x-direction

$$-2kx\cos^2\theta.$$

(d) (2 marks) This part involves four springs along the diagonals of a square, ABCD, as shown in figure 5.

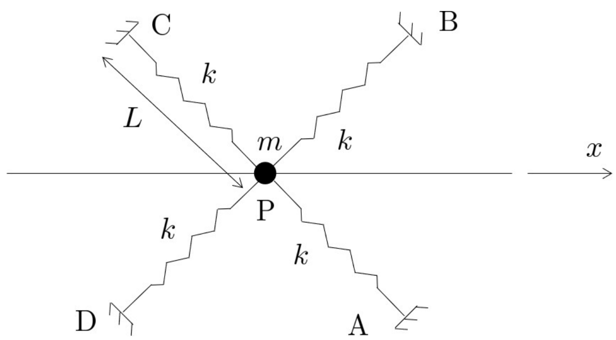


Figure 5: System with four springs along the diagonals of a square.

Find the value of λ_3.

(e) (1 mark) This final part involves six springs arranged in a regular hexagonal form, ABCDEF, as shown in figure 6.

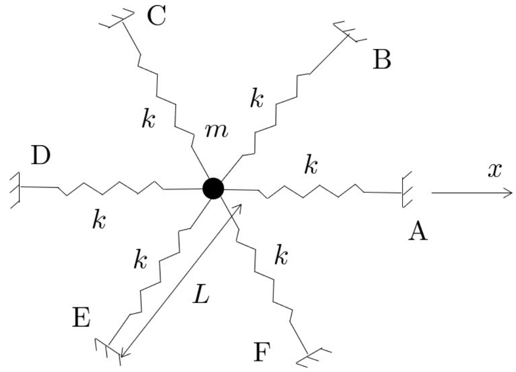


Figure 6: System with six springs in a regular hexagon.

Find the value of λ_4.

6 The Firefly Dance

A team of biologists on the space station are struggling to understand the patterns they see from the fireflies in their lab. It is up to you as physicists to show them biologists up and investigate the synchronization of fireflies.

Fireflies provide one of the most spectacular examples of synchronization in nature. Male fireflies gather on trees and flash on and off in unison at night in order to attract female fireflies.

A simple model of synchronization of fireflies' flashing rhythm in response to stimuli can be given as follows. Let $\theta(t)$ be the phase of the firefly's flashing rhythm, with $\theta = 0$ being the instant when a flash is emitted. Assume that in the absence of external stimuli, the firefly goes through this cycle of on and off at a constant frequency ω, so that $\dot{\theta} = \omega$, where $\dot{\theta} = d\theta/dt$.

Now suppose there exists an external periodic stimulus whose phase Θ satisfies the same differential equation but different frequency Ω, i.e.

$$\dot{\Theta} = \Omega\,, \tag{1}$$

where $\Omega = 0$ is when the flash of the external stimulus is on. This stimulus can be e.g. another firefly, or even artificial LED light. We can model the firefly's response to this stimulus as follows:

- If the stimulus is ahead in the cycle, then we assume that the firefly *speeds up* in attempt to synchronize.
- Conversely, the firefly tries to slow down if the stimulus is behind (i.e. firefly is flashing too early).

A simple model that incorporates these assumptions is

$$\dot{\theta} = \omega + A\sin(\Theta - \theta)\,, \qquad A > 0\,. \tag{2}$$

Clearly, if Θ is ahead of θ, i.e. $\Theta - \theta \in (0, \pi)$ then the firefly speeds up ($\dot{\theta} > \omega$). The parameter A is called the *resetting strength* of the firefly, which measures the ability of the firefly to modify its instantaneous frequency.

We say that the firefly has been **entrained** by the stimulus if it successfully matches the frequency of the stimulus.

(a) (1 mark) By defining two dimensionless quantities τ (dimensionless time) and μ, construct a *single* differential equation for a function ϕ that describes the dynamics of synchronization given by the model above.

What quantity is ϕ? (**Hint:** two words)

(b) (2 marks) Sketch the phase diagram (ϕ', ϕ), where $\phi' = d\phi/d\tau$ for the three cases: (i) $\mu = 0$, (ii) $0 < \mu < 1$, and (iii) $\mu > 1$. On the ϕ-axis of each diagram, draw an arrow to the right if $\phi' > 0$ and to the left when $\phi' < 0$. Hence, show that (i) has three fixed points, one of which is stable, (ii) has one stable and one unstable fixed points, while (iii) has no fixed point at all.

(c) (1 mark) Use the results so far to argue that for $\mu = 0$ the firefly will eventually flash *simultaneously.*

(d) (1 mark) In the similar spirit as part (c), argue that for $\mu \in (0, 1)$ the firefly will be *phase-locked* to the stimulus. Your answer should explain why the term "phase-locking" is used in this regime and the nature of the phase locking itself.

(e) (1 mark) For $\mu > 1$, the firefly is said to be *phase-drifted* relative to the stimulus. However, the phase drift is not uniform. When is the phase drift, the fastest and the slowest in terms of the phases of the firefly and the stimulus?

(f) (2 marks) The model makes a testable and specific predictions. Show that the **range of entrainment**, which is the range of frequency of the firefly flashing for which it can be entrained, is given by

$$\omega - A \leq \Omega \leq \omega + A\,.$$

The parameter A is typically determined by experiments.

(g) (2 marks) For $\mu > 1$, calculate the period of the phase drift T, which is the time taken for the stimulus and the firefly to flash simultaneously again (but they never synchronize). Hence, explain physically why this period is ill-defined or should be regarded as infinity for $\mu \in [0, 1)$.

7 Bending Light in a Plane

One of the concerns with moving the space station is that it might end up in a black hole and transport us to a different reality that exists only in two-dimensions. With concerns mounting, you have been tasked to investigate how light would behave in this two-dimensional world.

Imagine living in a (two-dimensional) world where the refractive index varies as a function of height:

$$n(y) = \sqrt{1 + e^{-y/2}}$$

A person in this world stands at the origin ($x = 0, y = 0$) and shines a light beam directed at 45° upward.

(a) (9 marks) Determine the equation of the path of the light beam. (Hint: light always travel in the path that takes the least time.)

(b) (1 mark) Plot the path of the light beam.

8 A Magnetic Spa Day

Researchers at Cheesecake are developing on the concept of a super-efficient magnet-based engine. For these purposes, they study a magnetic dipole-dipole interaction in a hot environment.

Suppose $\mathbf{m}_1$ and $\mathbf{m}_2$ are two classical dipole moments, separated by a vector $\mathbf{r}$. The potential energy from the dipole-dipole interaction is

$$V(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\left[\mathbf{d}_1 \cdot \mathbf{d}_2 - 3(\mathbf{d}_1 \cdot \hat{\mathbf{r}})(\mathbf{d}_2 \cdot \hat{\mathbf{r}})\right]$$

where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$.

(a) (2 marks) By parameterising the vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ in spherical polar coordinates (d_i, θ_i, ϕ_i) in the usual way and choosing a coordinate system where $\mathbf{r}$ is in the z direction, show that the potential may be parameterised as

$$V(r) = \frac{\mu_0 d_1 d_2}{4\pi r^3}\left(\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) - 2\cos\theta_1 \cos\theta_2\right).$$

(b) (7 marks) Suppose the dipoles are both in thermal equilibrium with a heat bath at temperature T, and consider the limit of large temperature where

$$\lambda = \frac{\mu_0 d_1 d_2}{4\pi r^3 k_B T} \ll 1$$

Calculate the mean force between the dipoles in the z direction to leading order in λ. Is this force attractive or repulsive?

(c) (1 mark) In the same limit as above, explain why the mean force in the perpendicular directions will be zero.

9 Droplets in SPACE

You are conducting some experiments on liquid droplets in Cheesecake's 'zero gravity' lab. After work, you crack open a few drinks with your mates. When Tigger spilt his drink, you realised that he'd had a bit much, but you also noticed droplets starting to form in thin air coming from the spilt drink.

In this task, you are not expected to know exact expressions for the capillary force for specific geometry. You rather should base your derivation on the logic of important parameters measurement units.

A free-floating spherical droplet (in the absence of gravity) can experience deformation oscillations, which are driven by the surface tension forces and liquid inertia.

(a) (5 marks) Your task is to estimate the main mode eigenfrequency of the droplet oscillation.

The same liquid being evaporated condensates at the space station's ceiling and form a group of growing droplets. Consider the balance between surface tension force, which keeps the droplet in place, and gravity force, which pulls the droplet down.

(b) (5 marks) Your task is to estimate the maximal possible radius of the droplet which stays on the ceiling.

10 Oscillators on a Train

While on the train going to give your report on all previous investigations, you take the opportunity to conduct one more investigation with your pocket oscillator.

You are sitting on a train, and have an oscillator which consists of a point mass m moving in a harmonic well $V(x) = kx^2/2$ in the reference frame of the train. The period of the oscillator T is exactly one second; and its orientation coincides with the direction of motion of the train (i.e. the train will also move in the x direction). The train starts at rest in the station, and will reach its cruising speed of 30 metres per second by a constant acceleration over exactly one minute. For this scenario, you are to take the mass of the point mass m to be exactly *one standard Anthony.*

(a) (2 marks) Calculate the period of the oscillator during the accelerating period of the journey.

(b) (4 marks) While sitting in the station, you place the point mass at rest at the bottom of the harmonic well $x = 0$. When the train leaves the station and starts to accelerate, the mass will start to move, undergoing oscillations. Calculate the amplitude of these oscillations.

(c) (3 marks) Calculate the amplitude of oscillations after the train reaches its cruising speed.

(d) (1 mark) Does your answer to the last part depend only on the final speed reached, or does it also depend on the acceleration used to get there? Briefly discuss why this makes sense.

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